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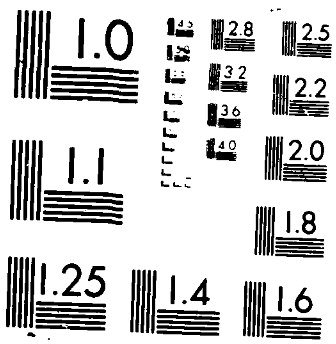

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DETECTING AND INTERVAL ESTIMATION ABOUT A SLOPE CHANGE POINT*

P. R. Krishnaiah and B. Q. Miao

ABSTRACT

In this paper, the authors consider the problem of change points using Gaussian process. The distribution of the statistic to estimate a change point constructed in this paper can be approximated by the first type of extrimal distribution. Based on this, detection and interval estimation of a change point in various situations are discussed.

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1. INTRODUCTION

Consider the model

$$x(t) = f(t) + e_t, \quad 0 < t \leq 1 \tag{1.1}$$

where f(t) is a nonrandom function with the form:

$$f(t) = \begin{cases} \mu + \beta_1(t - t_0), & 0 < t \leq t_0 \\ \mu + \beta_2(t - t_0), & t_0 < t \leq 1. \end{cases} \tag{1.2}$$

t_0 is called the slope change point (of the function f(t)). $\{e_t, 0 < t \leq 1$ is an independent random process with zero mean function.

In order to estimate and make inference on t_0, observe x(t) in equal space, that is, we observe $x(\frac{i}{n})$, $i = 1,2,\ldots,n$. For simplicity, we write x_i and e_i for $x(\frac{i}{n})$ and $e(\frac{i}{n})$, respectively, but we must keep in mind that x_i and e_i are dependent on i and n, and $e_1, \ldots, e_n$ are independent. Generally, μ, β_2 and β_1 are unknown. There are many ways to estimate the location of t_0. For example, see Hudson (1966), Hinkley (1970) and Krishnaiah and Miao (1986a, 1986b), but it is more important to make an interval estimate of t_0. This problem is associated with the distribution of the estimator to t_0. Feder (1975) proved that the LSE (Least Square Estimator) of t_0 is asymptoticly normal. Hinkley (1971) proposed an approximate distribution of the MLE (Maximum Likelihood Estimator) of t_0, but it is too complex. If t_0 is the jump-point, Csörgö and Horváth (1986) proposed some asymptotic distributions for some nonparamatric estimators of t_0.

Recently, Chen (1987) developed such an estimator of t_0 where distribution is the first type of extrimal distribution. This estimator of t_0 is proposed first by Yin (1986) to estimate the location of one or more

change points. Going along with this heuristic method, we give an estimator of t_0 for models (1.1) and (1.2). Its distribution can then be calculated conveniently.

In Section 2 we treat the case that $e_1, \ldots, e_n$ are normal with zero mean and positive variance σ^2. In Section 3 we treat the case that $e_1, \ldots, e_n$ are normal with zero mean, but their common variance is unknown. When random errors $e_1, e_2, \ldots$ are not normal, for example, e_i has moment generating function, or only has finite $(2+\delta)$-th moment, the conclusion established in Section 1 is also true. This is discussed in Section 4. Finally, in Section 5 we discuss the estimation of the slope change $\beta_1 - \beta_2$, under some mild conditions. This estimation is asymptotically normal.

2. ERROR IS NORMAL WITH A KNOWN VARIANCE

In this section we suppose that $\{e(t)\}$ is a white noise process with mean zero and known variance σ^2. At first we prove a theorem on which our method is based.

THEOREM 1. Suppose that

$$x_k = a + \frac{k}{n}\beta + \varepsilon_k, \quad k = 1, \dots, n, \tag{2.1}$$

where $\varepsilon_1, \dots, \varepsilon_n$ are i.i.d., $\varepsilon_1 \sim N(0,\sigma^2)$. Let $m = m_n$ be a positive integer such that

$$n \gg m \gg n^{2/3}\log^{2/3} n. \tag{2.2}$$

Hereafter, $u_n \gg v_n > 0$ means $\lim_{n\to\infty} \frac{u_n}{v_n} = \infty$. Set

$$Y_k = \frac{1}{2\sqrt{m}}\Big[(x_{k-4m+1} + \dots + x_{k-3m}) - (x_{k-3m+1} + \dots + x_{k-2m})$$
$$- (x_{k-2m+1} + \dots + x_{k-m}) + (x_{k-m+1} + \dots + x_k)\Big],$$
$$k = 4m, 4m+1, \dots, n. \tag{2.3}$$

Write

$$\xi_n = \max_{4m \le k \le n} |Y_k|,$$

and

$$A_n(x) = \left(2\log\left(\frac{5n}{4m} - 5\right)\right)^{-1/2}$$
$$\left(x + 2\log\left(\frac{5n}{4m} - 5\right) + \frac{1}{2}\log\log\left(\frac{5n}{4m} - 5\right) - \frac{1}{2}\log\pi\right), \tag{2.4}$$

Then

$$\lim_{n\to\infty} P\left(\frac{\xi_n}{\sigma} \leq A_n(x)\right) = \exp\{-2e^{-x}\}, \quad -\infty < x < \infty. \tag{2.5}$$

Proof. Construct a standard Brownian Motion $\{W(t): t \geq 0\}$, such that

$$W(\tfrac{5k}{4m}) = \sqrt{\frac{5}{4m}}\left(x_1 + \dots + x_k - ka - \frac{k(k+1)}{2n}\beta\right)/\sigma, \quad k = 4m,\dots,n. \tag{2.6}$$

Based on this $W(t)$, we further construct the Gaussian process $Z(t)$ such that

$$Z(t) = \frac{1}{\sqrt{5}}\left[W(t+5) - 2W(t+\tfrac{15}{4}) + 2W(t+\tfrac{5}{4}) - W(t)\right], \quad t \geq 0. \tag{2.7}$$

It is easy to see that

$$Y_k = \sigma Z(\tfrac{5k}{4m} - 5), \quad k = 4m,\dots,n, \tag{2.8}$$

and the covariance function $\rho(\tau)$ of $Z(t)$ is

$$\rho(\tau) = \begin{cases} 1 - |\tau| & |\tau| \leq \frac{5}{4} \\ -\frac{1}{5}|\tau| & \frac{5}{4} \leq |\tau| \leq \frac{5}{2} \\ \frac{3}{5}|\tau| - 2 & \frac{5}{2} \leq |\tau| \leq \frac{15}{4} \\ 1 - \frac{1}{5}|\tau| & \frac{15}{4} \leq |\tau| \leq 5 \\ 0 & |\tau| > 5 \end{cases}. \tag{2.9}$$

Set

$$\tilde{\xi}_n = \sup\{|Z(t)|: 0 \leq t \leq \tfrac{5n}{4m} - 5\},$$

$$\eta_n = \tilde{\xi}_n - \sigma\xi_n.$$

It can be proved, similar to Chen's method, that

$$\lim_{n\to\infty} \eta_n \sqrt{\log n} = 0, \quad \text{a.s.} \tag{2.10}$$

For the Gaussian process $Z(t)$ with covariance $\rho(\tau)$, the conditions of a theorem of Qualls and Watanable (1972) are satisfied, we get

$$\lim_{n\to\infty} P\left(\tilde{\xi}_n \le A_n(x)\right) = \exp\{-2e^{-x}\}. \tag{2.11}$$

But $A_n(x)$ is a linear function of x, hence for n large,

$$P\left(\tilde{\xi}_n \le A_n(x - |\Delta x|)\right) - P(\eta_n \ge |\Delta x|/\sqrt{2\log n}) \le P\left(\xi_n/\sigma \le A_n(x)\right)$$
$$\le P\left(\tilde{\xi}_n \le A_n(x + |\Delta x|)\right) + P(\eta_n \ge |\Delta x| \sqrt{2\log n}). \tag{2.12}$$

From (2.10) to (2.12), letting $n \to \infty$, then $\Delta x \to 0$, we get this theorem.

This theorem represents an asymptotic distribution of statistic ξ_n. It suggests a way to test the null hypothesis:

$$H_0: \theta = 0, \tag{2.13}$$

i.e., there is no slope change point in model (1.1) and (1.2), as follows. For the chosen level α, $0 < \alpha < 1$, solving the equation $\exp(-2e^{-x}) = 1 - \alpha$, we get $x(\alpha) = -\log\left(-\frac{1}{2}\log(1-\alpha)\right)$. Set

$$d = \frac{4m}{n}, \qquad C_n(\alpha,d) = A_n\left(x(\alpha)\right). \tag{2.14}$$

The null hypothesis (2.13) is rejected when and only when

$$\xi_n > \sigma C_n(\alpha,d). \tag{2.15}$$

Under the hypothesis (2.13), this test has an asymptotic level α as sample size n tends to infinity.

Next we also give an estimate of the power $\beta_n = \beta_n(\beta_1, \beta_2, \sigma)$ of this test. Let r be the integer such that

$$\frac{r}{n} \le t < \frac{r+1}{n} .$$

Then

$$Y_{r+2m} \sim N\left(\frac{m^{3/2}}{2n}(\beta_2 - \beta_1),\ \sigma^2\right).$$

Hence,

$$\beta_n(\beta_1, \beta_2, \sigma) \ge P\left(|Y_{r+2n}| > \sigma C_n(\alpha,d)\right)$$

$$> \Phi\left(\frac{m^{3/2}}{2n\sigma}|\beta_2 - \beta_1| - C_n(\alpha,d)\right) \qquad (2.16)$$

where

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}dt.$$

From this inequality, in order to get a larger β_n, m must be given a larger value. Note that the order of $C_n(\alpha,d)$ is $(\log \frac{n}{m})^{1/2}$ for fixed α, so our test has a larger power when and only when $m >> n^{2/3}\log^{2/3}n$. It is very differe from the case that f(t) is a step function. In our case, the power is lower. The reason is evident because there exist more estimated parameters.

Now consider the interval estimation of a slope change point. The existence of t_0 may be a fact known in advance, but usually it is evidenced by the rejection of the null hypothesis. If t_0 is evidenced to exist, we adopt the following rule.

RULE. Find an integer k such that $|Y_k| = \xi_n$. Take $\left[\frac{k-4m}{n}, \frac{k}{n}\right]$ as the confidence interval of t_0.

The length of this interval is $\frac{4m}{n}$. Hence, the smaller the value of

m, the more accurate is the estimate. But, as described by Chen, m cannot be taken too small so as to get a high confidence coefficient and decrease the risk of false acceptance of the hypothesis (2.13) if the existence of t_0 is to be decided by the test above. Here we give an estimate of the confidence coefficient γ of this interval as follows.

$$\gamma = P\left(\frac{k-4m}{n} \le t_0 \le \frac{k}{n}\right)$$
$$\ge P\left(\left\{\sup_{k\notin[r,r+4m]} |Y_k| \le \sigma C_n(\alpha,d)\right\} \cap \left\{|Y_{r+2m}| > \sigma C_n(\alpha,d)\right\}\right).$$

Set

$$A = \left\{\sup_{4m\le k<r} |Y_k| \le \sigma C_n(\alpha,d)\right\},$$

$$B = \left\{\sup_{r+4m<k\le n} |Y_k| \le \sigma C_n(\alpha,d)\right\},$$

$$B_1 = \left\{\sup_{r+6m<k\le n} |Y_k| \le \sigma C_n(\alpha,d)\right\},$$

and

$$C = \left\{|Y_{r+2m}| > \sigma C_n(\alpha,d)\right\}.$$

Notice that B_1 is independent of both A and C, and $B\subset B_1$, we have

$$\gamma \ge P\big((A\cup B)C\big) = P(AC) + P(\bar{A}BC) = P(C) + P(B) - P(\bar{A}C\cup B)$$
$$\ge P(C) + P(B) - P(\bar{A}C\cup B_1)$$
$$\ge P(C) + P(B) - P(\bar{A}C) - P(B_1) + P(\bar{A}B_1C)$$
$$= P(C) + P(B) - P(\bar{A}C) - P(B_1) + P(\bar{A}C)P(B_1)$$
$$\ge P(C) - \big(P(B_1) - P(B)\big) - P(\bar{A})P(\bar{B}_1),$$

where $\bar{D}$ denotes the complementary event of D. Again, using Theorem 1, we get

$$\gamma \geq \Phi\left(\frac{m^{3/2}|\beta_2-\beta_1|}{2n\sigma} - C_n(\alpha,d)\right)$$
$$- \left(\exp\{-2e^{-x_3(\alpha)}\} - \exp\{-2e^{-x_2(\alpha)}\}\right)$$
$$-\left(1 - \exp\{-2e^{-x_1(\alpha)}\}\right)\left(1 - \exp\{-2e^{-x_3(\alpha)}\}\right) \tag{2.17}$$

where

$$x_1 = x_1(\alpha) = C_n(\alpha,d)\left(2 \log(\tfrac{5r}{4m} - 5)\right)^{1/2}$$
$$- \left(2 \log(\tfrac{5r}{4m} - 5) + \tfrac{1}{2} \log \log(\tfrac{5r}{4m} - 5) - \tfrac{1}{2} \log \pi\right), \tag{2.18}$$

$$x_2 = x_2(\alpha) = C_n(\alpha,d)\left(2 \log(\tfrac{5(n-r)}{4m} - 5)\right)^{1/2}$$
$$- \left(2 \log(\tfrac{5(n-r)}{4m} - 5) + \tfrac{1}{2} \log \log(\tfrac{5(n-r)}{4m} - 5) - \tfrac{1}{2} \log \pi\right), \tag{2.19}$$

and

$$x_3 = x_3(\alpha) = C_n(\alpha,d)\left(2 \log(\tfrac{5(n-r)}{4m} - 7.5)\right)^{1/2}$$
$$- \left(2 \log(\tfrac{5(n-r)}{4m} - 7.5) + \tfrac{1}{2} \log \log(\tfrac{5(n-r)}{4m} - 7.5) - \tfrac{1}{2} \log \pi\right). \tag{2.20}$$

As a rough approximation, if we have no information about t_0, applying this fact that

$$P\left(\sup_{k \notin [r,r+4m]} |Y_k| \leq \sigma C_n(\alpha,d)\right) \geq P\left(\sup_{4m \leq k \leq n} |Y_k| \leq \sigma C_n(\alpha,d)\right)$$
$$\approx 1 - \alpha, \tag{2.21}$$

we get

$$\gamma > \Phi\left(\frac{m^{3/2}|\beta_2-\beta_1|}{2n\sigma} - C_n(\alpha,d)\right) - \alpha. \tag{2.22}$$

By those inequalities above, we see that γ is larger as $\frac{m^{3/2}|\beta_2-\beta_1|}{2n\sigma}$ is larger. Note that $\left|\frac{k\beta_2}{n}-\frac{k\beta_1}{n}\right|$ is the absolute of difference between $f(\frac{k}{n}+t_0) - f(t_0)$ and $f(t_0) - f(t_0-\frac{k}{n})$. Now the length of confidence interval is $\frac{4m}{n}$, the slope change point t_0 is of practical means only when $\frac{m}{n}|\beta_2-\beta_1|$ is larger than σ. Generally, we can assume that $\frac{m}{n\sigma}|\beta_2-\beta_1| \geq M$, where M is decided by practical consideration.

Using (2.17) or (2.22), we may give the following important question an estimation on the integers m and n: form a confidence interval of t_0 with prescribed length d_0 and confidence coefficient $1 - \alpha_0$. To do this, if there is no information on t_0, we solve this equation by replacing α and d in (2.22) by $\alpha_0/2$ and d_0,

$$\Phi\left(\frac{M}{2}\sqrt{m} - C_n(\alpha_0/2, d_0)\right) - \alpha_0/2 = 1 - \alpha_0,$$

and get

$$m \approx 4M^{-2}\left(C_n(\alpha_0/2, d_0) + u_{\alpha_0/2}\right)^2, \tag{2.23}$$

and

$$n \approx \frac{4m}{d_0}, \tag{2.24}$$

where $M = \frac{m|\beta_2-\beta_1|}{n\sigma}$, $u_{\alpha_0/2}$ is the number such that $1 - \Phi(u_{\alpha_0/2}) = \alpha_0/2$. For example, let $M = 3$, $d_0 = 0.1$ and $\alpha_0 = 0.05$. Then

$$m \approx 18, \qquad n \approx 714.$$

If we further know that $an < t_0 < bn$, here a and b are constants known a priori, then by (2.17) we could solve the equation:

$$\Phi\left(\frac{M}{2}\sqrt{m} - C_n(\alpha, d_0)\right) - \left(\exp\{-2e^{-x_3(\alpha)}\} - \exp\{-2e^{-x_2(\alpha)}\}\right)$$
$$- \left(1 - \exp\{-2e^{-x_1(\alpha)}\}\right)\left(1 - \exp\{-2e^{-x_3(\alpha)}\}\right) = 1 - \alpha_0. \quad (2.25)$$

For example, let $M = 3$, $d_0 = 0.1$, $\alpha = \alpha_0 = 0.05$ and $a = 0.2$, $b = 0.8$. Then,

$$C_n(0.05,\ 0.1) = 4.1217$$

$$m \approx 15, \qquad n \approx 597.$$

Based on the results above, we see that if more information about t_0 is known, then not only does it increase the confidence coefficient of γ, but also decrease the threshold value of rejecting the null hypothesis (2.13).

Table: The values of (m,n) when $r \leq t_0 < i - r$,
$M = 3$, $\alpha_0 = 0.05$ and $d_0 = 0.1$.

α \ r (m,n)	0	0.1	0.15	0.2
0.05		20.69, 828	14.93, 597	14.92, 597
0.025	17.85, 714	17.70, 708	16.18, 647	16.17, 647

3. ERROR IS NORMAL WITH UNKNOWN VARIANCE

When σ^2 is unknown, we can use its estimate, say $\hat{\sigma}_n^2$. Then substituting $\hat{\sigma}_n$ for σ in (2.15) to perform the test, Chen proved the following theorem.

THEOREM 2. Under the conditions of Theorem 1, if $\hat{\sigma}_n^2$ is an estimator of σ^2 satisfying

$$\lim_{n\to\infty}|\hat{\sigma}_n^2 - \sigma^2|\log n \overset{P}{=} 0, \tag{3.1}$$

"$\overset{P}{=}$" means convergence in probability. Then

$$\lim_{n\to\infty} P\left(\xi_n/\hat{\sigma}_n - A_n(x)\right) = \exp\{-2e^{-x}\}.$$

Our problem is to find such an estimator satitying (3.1). The LSE of σ^2 suggests the form (3.5) given below. We prove this estimator satisfies (3.1).

Suppose $(x_1, \ldots, x_n)$ is observed from the model (1.1) and (1.2). Then

$$x_i = \begin{cases} \mu_1 + \dfrac{i-n_1}{n}\beta_1 + \varepsilon_i, & i = 1,\ldots,n_1 \\[2ex] \mu_2 + \dfrac{i-n_1}{n}\beta_2 + \varepsilon_i, & i = n_1+1,\ldots,n. \end{cases} \tag{3.2}$$

where we assume that the slop change point t_0 falls into $[\frac{n_1}{n}, \frac{n_1+1}{n})$. By (1.2), we have

$$|\mu_1 - \mu_2| \le \frac{1}{n}|\beta_2 - \beta_1|. \tag{3.3}$$

Let

$$\bar{x}_{1c} = \frac{1}{c}\sum_{i=1}^{c} x_i, \qquad \bar{x}_{2m} = \frac{1}{n-c}\sum_{i=c+1}^{n} x_i,$$

$$\Sigma_{Lc} = \frac{2}{c(c-1)}\sum_{i=1}^{c}(c-i)x_i, \qquad \Sigma_{Rc} = \frac{2}{(n-c)(n-c+1)}\sum_{i=c+1}^{n}(i-c)x_i.$$

Then the following result holds:

THEOREM 3. If $\varepsilon_1, \ldots, \varepsilon_n$ are i.i.d., and $\varepsilon_1 \sim N(0,\sigma^2)$, set

$$S_{nc}^2 = \sum_{i=1}^{c}(x_i - \bar{x}_{1c})^2 + \sum_{i=c+1}^{n}(x_i - \bar{x}_{2c})^2 - \frac{3c(c-1)}{c+1}(\Sigma_{Lc} - \bar{x}_{1c})^2$$
$$- \frac{3(n-c)(n-c+1)}{n-c-1}(\Sigma_{Rc} - \bar{x}_{2c})^2. \tag{3.4}$$

$$\hat{\sigma}_{nc}^2 = \frac{1}{n}S_{nc}^2, \qquad c = m+1,\ldots,n-m. \tag{3.5}$$

Then

$$|\min_{m\le c\le n-m} \hat{\sigma}_{nc}^2 - \sigma^2|\log n \xrightarrow{P} 0. \tag{3.6}$$

Proof. It is easy to see that the expressions of (3.4) and (3.5) are the form of LSE of model (3.2) under the assumption that c is the slope change point. Write

$$F_c = \begin{pmatrix} e_c & -\frac{1}{n}f_c & 0 & 0 \\ 0 & 0 & e_{n-c} & \frac{1}{n}g_{n-c} \end{pmatrix} \tag{3.7}$$

$$e_j' = (1,\ldots,1)_{1\times j}', \qquad f_j' = (j-1, j-2, \ldots, 1, 0)_{1\times j}'$$
$$g_j' = (1,\ldots,j)_{1\times j}', \qquad \beta = (\mu_1, \beta_1, \mu_2, \beta_2)' \tag{3.8}$$
$$x = (x_1,\ldots,x_n)' \text{ and } \varepsilon = (\varepsilon_1,\ldots,\varepsilon_n)'.$$

Then

$$x = F_{n_1}\beta + \varepsilon \tag{3.9}$$

and

$$S_{nc}^2 = x'\left(I - F_c(F_c'F_c)^{-1}F_c'\right)x.$$

Our line to prove this theorem is as follows: When $\beta_1 \neq \beta_2$, let h be the integer such that $\hat{\sigma}_{nh}^2 = \min_{1\leq c\leq n} \hat{\sigma}_{nc}^2$.

1. If $|h-m| \geq n/\log^2 n$, then $S_{nc}^2 - S_{nn_1}^2 > 0$ in probability.

2. If $|h-m| \leq n/\log^2 n$, then $|\hat{\sigma}_{nc}^2 - \hat{\sigma}_{nn_1}^2|\log n \xrightarrow{P} 0$.

3. $|\hat{\sigma}_{nn_1}^2 - \sigma^2|\log n \xrightarrow{P} 0$.

When $\beta_1 = \beta_2$, we have for any c, $|\hat{\sigma}_{nc}^2 - \sigma^2|\log n \xrightarrow{P} 0$. It can be calculated that

$$F_c'F_c = \begin{pmatrix} c & -\frac{1}{n}\sum_{i=1}^{c-1} i & 0 & 0 \\ -\frac{1}{n}\sum_{i=1}^{c-1} i & \frac{1}{n^2}\sum_{i=1}^{c-1} i^2 & 0 & 0 \\ 0 & 0 & (n-c) & \frac{1}{n}\sum_{i=1}^{n-c} i \\ 0 & 0 & \frac{1}{n}\sum_{i=1}^{n-c} i & \frac{1}{n^2}\sum_{i=1}^{n-c} i^2 \end{pmatrix}. \tag{3.10}$$

$$(F_c'F_c)^{-1} = \begin{pmatrix} \frac{2(2c-1)}{c(c+1)} & \frac{6n}{c(c+1)} & 0 & 0 \\ \frac{6n}{c(c+1)} & \frac{12n^2}{c(c^2-1)} & 0 & 0 \\ 0 & 0 & \frac{2(2n-2c+1)}{(n-c)(n-c-1)} & \frac{-6n}{(n-c)(n-c-1)} \\ 0 & 0 & \frac{-6n}{(n-c)(n-c-1)} & \frac{12n^2}{(n-c)(n-c+1)(n-c-1)} \end{pmatrix}$$

$$\hat{=} \begin{pmatrix} a_{1c} & a_{2c}n & 0 & 0 \\ a_{2c}n & a_{4c}n & 0 & 0 \\ 0 & 0 & b_{1c} & -b_{2c}n \\ 0 & 0 & -b_{2c}n & b_{4c}n \end{pmatrix}, \tag{3.11}$$

$$F_c(F_c'F_c)^{-1}F_c' = \begin{pmatrix} a_{1c}e_ce_c' - a_{2c}f_ce_c' - a_{2c}e_cf_c' + \frac{a_4}{n}f_cf_c', & 0 \\ 0, & b_{1c}e_{n-c}e_{n-c}' - b_{2c}g_{n-c}e_{n-c}' - b_{2c}e_{n-c}g_{n-c}' + \frac{b_{4i}}{n}g_{n-c}g_{n-c}' \end{pmatrix}. \tag{3.12}$$

Not loss of generality, we assume that $n > c > n_1$. Set $k \hat{=} c - n_1$.

$$F_{c-n_1} \hat{=} F_c - F_{n_1} = \begin{pmatrix} 0 & -\frac{k}{n}e_{n_1} & 0 & 0 \\ e_{c-n_1} & -\frac{1}{n}f_{c-n_1} & -e_{c-n_1} & -\frac{1}{n}g_{c-n_1} \\ 0 & 0 & 0 & -\frac{k}{n}e_{n-c} \end{pmatrix}. \tag{3.13}$$

After omitting some 1 and -1, for example, replacing n_1 for n_1-1 or n_1+1, we get

$$(F_c - F_{n_1})'F_c(F_c'F_c)^{-1}F_c' = \begin{pmatrix} \frac{k}{c^3}(4c^2 - 9kc + 6k^2)e_{n_1}' - \frac{6k(c-k)}{c^3}f_{n_1}' & \frac{k}{c^3}\Big(c(4c-3k)e_{c-n_1}' - 6(c-k)f_{c-n_1}'\Big) & 0 \\ - \frac{k}{nc^3}\Big((c^3 - 2kc^2 + 4k^2c - 2k^3)e_{n_1}' + k(3c-2k)f_{n_1}'\Big) & - \frac{k}{nc^3}\Big(c(c-k)^2e_{c-n_1}' + k(3c-2k)f_{c-n_1}'\Big) & 0 \\ - \frac{k}{c^3}\Big((4c^2 - 9kc + 6k^2)e_{n_1}' - 6(c-k)f_{n_1}'\Big) & - \frac{k}{c^3}\Big(c(4c-3k)e_{c-n_1}' - 6(c-k)f_{c-n_1}'\Big) & 0 \\ - \frac{k^2}{nc^3}\Big((2(c-k)^2e_{n_1}' - (3c-2k)f_{n_1}'\Big) & - \frac{k^2}{n^3}\Big(c(2c-k)e_{c-n_1}' - (3c-2k)f_{c-n_1}'\Big) & -\frac{k}{c}e_{n-c}' \end{pmatrix}. \tag{3.14}$$

Set $G = F_c(F_c'F_c)^{-1}F_c' - F_{n_1}(F_{n_1}'F_{n_1})^{-1}F_{n_1}' = (g_{ij})_{n\times m}$. By complex calculation, we can get

$$E\left|\sum_{i=1}^{n} g_{ii}\varepsilon_i^2\right| \le E\sum_{i=1}^{n}\{tr\Big(F_c(F_c'F_c)^{-1}F_c'\Big) + tr\Big(F_{n_1}(F_{n_1}'F_{n_1})^{-1}F_{n_1}'\Big)\}\varepsilon_i^2 = 8\sigma^2, \tag{3.15}$$

$$E\left|\sum_{i\neq j} g_{ij}\varepsilon_i\varepsilon_j\right|^2 = \Sigma g_{ij}^2\sigma^4 \leq 280\sigma^4. \tag{3.16}$$

Write $\gamma' = \beta' F'_{c-n_1}\left(I - F_c(F_c'F_c)^{-1}F_c'\right)$. From (3.14) and (3.3), we get

$$\frac{k^4 n_1^3}{4n^2c^4}(\beta_2 - \beta_1)^2 \leq \gamma'\gamma \leq \frac{3k^4n_1^3}{n^2c^4}(\beta_2 - \beta_1)^2 + \frac{100k^2(n-c)}{n^2}\beta_2^2. \tag{3.17}$$

Hence,

$$\mathrm{Var}(\gamma'\varepsilon) = \sigma^2 \mathrm{tr}\ \gamma\gamma' = \sigma^2\gamma'\gamma \leq \frac{3k^4n_1^3}{n^2c^4}(\beta_2 - \beta_1)^2 + \frac{100k^2(n-c)}{n^2}\beta_2^2. \tag{3.18}$$

By (3.8), (3.9) and (3.5)

$$\hat{\sigma}_{nc}^2 - \hat{\sigma}_{nn_1}^2 = -2\gamma'\varepsilon + \varepsilon'G\varepsilon + \gamma'\gamma. \tag{3.19}$$

Now we discuss $(\hat{\sigma}_{nc}^2 - \hat{\sigma}_{nn_1}^2)$.

Case 1. $\beta_1 \neq \beta_2$ and $k = c - n_1 \geq \dfrac{n}{\log^2 n}$. We have

$$P\left(\hat{\sigma}_{nc}^2 - \hat{\sigma}_{nn_1}^2 \geq \frac{\gamma'\gamma}{2n}\right) = P\left(-2\gamma'\varepsilon + \varepsilon'G\varepsilon \geq -\frac{\gamma'\gamma}{2}\right)$$

$$\leq P(|\gamma'\varepsilon| \geq \gamma'\gamma/8) + P\left(|\varepsilon'G\varepsilon| \geq \frac{\gamma'\gamma}{4}\right)$$

$$\leq \frac{64}{(\gamma'\gamma)^2}\mathrm{Var}(\gamma'\varepsilon) + P\left(|(\mathrm{tr}\, G)|\varepsilon'\varepsilon \geq \frac{\gamma'\gamma}{8}\right) + P\left(\left|\sum_{i\neq j} g_{ij}\varepsilon_i\varepsilon_j\right| \geq \frac{\gamma'\gamma}{8}\right)$$

$$\leq \frac{64}{(\gamma'\gamma)^2}\mathrm{Var}(\gamma'\varepsilon) + \frac{8}{\gamma'\gamma}E\left|\sum_{i=1}^n g_{ii}\varepsilon_i^2\right| + \frac{64}{(\gamma'\gamma)^2}E\left(\sum_{i\neq j} g_{ij}\varepsilon_i\varepsilon_j\right)^2.$$

By (3.15)-(3.18),

$$P(\hat{\sigma}^2_{nc} - \hat{\sigma}^2_{nn_1} \geq \frac{\gamma'\gamma}{2n}) \geq \frac{64\sigma^2}{\gamma'\gamma} + \frac{64}{\gamma'\gamma}\sigma^2 + \frac{64\times 280\sigma^4}{(\gamma'\gamma)^2}$$

$$\leq K\cdot\frac{128\sigma^2(\beta_1-\beta_2)^{-2}}{k^4 n_1^3}n^2c^4 + 33600\sigma^4(\frac{n^2c^4}{k^4n_1^3})(\beta_2-\beta_1)^{-4}$$

$$\leq 128\sigma^2(\beta_2-\beta_1)^{-2}\begin{cases} \frac{n^2}{n_1^3} + 300\sigma^2(\beta_2-\beta_1)^{-2}(\frac{n^2}{n_1^3})^2, & \text{if } k \geq n_1 \\ \frac{n^2}{k^4} + 300\sigma^2(\beta_2-\beta_1)^{-2}(\frac{n^2}{k^4})^2, & \text{if } k \geq n_1 \end{cases}$$

$$< 130\sigma^2(\beta_2-\beta_1)^{-2}(\log n)^{-2} \to 0. \tag{3.20}$$

Case 2. $\beta_1 \neq \beta_2$, $k = c - n_1 < n/\log^2 n$. It follows that for any $u > 0$,

$$P(|\hat{\sigma}^2_{nc} - \hat{\sigma}^2_{nm}| \geq \frac{u}{\log n}) \leq P(|{-2}\gamma'\varepsilon + \varepsilon'G\varepsilon| \geq \frac{un}{2\log n}$$

$$\leq P(|2\gamma'\varepsilon| \geq \frac{un}{4\log n}) + P(|\varepsilon'G\varepsilon| \geq \frac{un}{4\log n})$$

$$\leq \frac{64\log^2 n}{u^2n^2}\cdot\gamma'\gamma\sigma^2 + \frac{4\log n}{un}\cdot 8\sigma^2 + \frac{280\sigma^4}{\tau^2\log^2 n} \to 0, \tag{3.21}$$

by (3.15)-(3.18).

Besides, note that $\sum_{i=1}^n(\varepsilon_i^2 - \sigma^2)$ is a martingale and $A_{n_1}(A'_{n_1}A_{n_1})^{-1}A'_{n_1} \geq 0$, so by Marcinkiewicz-Zygmund-Burkholder's martingale inequality, we have, for any τ, δ and u: $0 < \tau < \delta/(1+\delta)$, $u > 0$.

$$P(|\hat{\sigma}^2_{nm} - \hat{\sigma}^2| \geq un^{-\tau})$$

$$\leq P(|\varepsilon'\varepsilon - n\sigma^2| \geq \frac{un^{1-\tau}}{2}) + P\left(\varepsilon'A_{n_1}(A'_{n_1}A_{n_1})^{-1}A'_{n_1}\varepsilon \geq \frac{un^{1-\tau}}{2}\right)$$

$$\leq c_{\delta,u}E|\varepsilon_1|^{2+\delta}n^{-(1+\delta)(1-\tau)}\cdot n + P\left(tr\left(A_{n_1}(A'_{n_1}A_{n_1})^{-1}A'_{n_1}\right)\varepsilon'\varepsilon \geq \frac{un^{1-\tau}}{2}\right)$$

$$\leq c_{\delta,u}E|\varepsilon_1|^{2+\delta}n^{-(\delta-(1+\delta)\tau)} + \frac{n^\tau\cdot n\sigma^2}{u(n_1+1)(n-n_1+1)} \to 0. \tag{3.22}$$

From Case 1 and Case 2, the theorem is true when $\beta_1 \neq \beta_2$.

Case 3. $\beta_1 = \beta_2$. In this case, for any $u > 0$,

$$P(|\hat{\sigma}^2_{n_c} - \hat{\sigma}^2_{n_0}| \geq \frac{u}{\log n})$$

$$= P\{|x'\left(I - F_c(F_c'F_c)^{-1}F_c'\right)x - x'\left(I - F_0(F_0'F_0)^{-1}F_0'\right)x| \geq \frac{u}{\log n}\}.$$

Set

$$\gamma_0' = \beta' F_{c-0}'\left(I - F_c(F_c'F_c)^{-1}F_c'\right), \qquad G_0 = F_c(F_c'F_c)^{-1}F_c' - F_0(F_0'F_0)^{-1}F_0'.$$

Then

$$P(|\hat{\sigma}^2_{n_c} - \hat{\sigma}^2_{n_0}| \geq \frac{u}{\log n}) \leq P\{|2\gamma_0'\varepsilon| \geq \frac{un}{\log n}\} + P\{|\varepsilon' G_0\varepsilon)| \geq \frac{un}{\log n}\}.$$

But

$$P(|2\gamma_0'\varepsilon| \geq \frac{un}{\log n}) \leq \frac{4\sigma^2 \log^2 n}{u^2 n^2} \cdot tr(\gamma_0'\gamma)$$

$$\leq \frac{4\sigma^2 n \log^2 n}{u^2 \cdot n^2} \to 0, \quad (n \to \infty) \tag{3.23}$$

when $n - c \geq \log^2 n$,

$$P\{\varepsilon' G_0 \varepsilon \geq \frac{un}{\log n}\}$$

$$\leq P\{tr\left(F_c(F_c'F_c)^{-1}F_c' + F_0(F_0'F_0)^{-1}F_0'\right)\varepsilon'\varepsilon \geq \frac{un}{\log n}\}$$

$$\leq \frac{\log n}{un} \cdot \frac{n}{(c+1)(n-c+1)} \leq \frac{2}{u \log n} \to 0, \quad (n \to \infty). \tag{3.24}$$

When $n - c \leq \log^2 n$, going along with the same line as Case 2, we can also get

$$P(|\varepsilon' G_0 \varepsilon| \geq \frac{un}{\log n})$$

$$\leq P\{|\Sigma g_{ii}\varepsilon_i^2| \geq \frac{un}{2\log n}\} + P\{|\sum_{i \neq j} g_{ij}\varepsilon_i\varepsilon_j| \geq \frac{un}{2\log n}\}$$

$$\to 0, \quad (n \to \infty). \tag{3.25}$$

By (3.23)-(3.25), we have

$$\hat{\sigma}^2_{n_c} - \hat{\sigma}^2_{n_0} \xrightarrow{P} 0, \quad \text{as } n \to \infty.$$

Finally, going along with the same line as (3.22), for any $u > 0$ and τ such that $0 < \tau < \frac{\delta}{1+\delta}$, we have

$$P(|\hat{\sigma}^2_{n_0} - \sigma^2| \geq un^{-\tau})$$

$$\leq P(|\varepsilon'\varepsilon - n\sigma^2| \geq \frac{un^{1-\tau}}{2}) + P(\varepsilon' F_0(F_0'F_0)^{-1}F_0\varepsilon \geq \frac{un^{1-\tau}}{2})$$

$$\leq c_\delta \cdot \frac{n^{\tau(1+\delta)}}{n^\delta} + c_\delta \frac{n^\tau}{n} \cdot 2\sigma^2 \to 0, \quad (\text{as } n \to \infty), \tag{3.26}$$

where c_δ is a constant only dependent upon ε.

Thus we complete the proof.

4. WHEN ERROR IS NON-NORMAL

When the distribution of random error $e(t)$ is nonnormal, we can use the theory of strong approximation of partial sums of i.i.d. variables by Brownian Motion Process to give some extensions of Theorem 1 to nonnormal errors.

THEOREM 4. Let $e_1, e_2, \ldots$ be i.i.d. random errors, and their common moment generating function exists in a small neighborhood of zero, i.e.,

$$E \exp(te_1) < \infty \quad \text{for} \quad |t| \text{ small enough,} \tag{4.1}$$

then the result of Theorem 1 remains valid.

Proof. Put

$$S_k \hat{=} S_{nk} = \sum_{i=1}^{k} (x_i - a - \frac{i}{n}\beta)/\sigma, \quad k = 1,2,\ldots,n,$$

then there exists a Brownian motion process $\{W(t), t \geq 0\}$ such that

$$\lim_{n\to\infty} \sup\{\sup_{k\leq n}|S_k - W(k)|/\log n\} < \infty, \quad \text{a.s.} \tag{4.2}$$

based on Komlós-Major-Tusnády (1975, 1976).

Since

$$\frac{Y_k}{\sigma} = \frac{1}{2\sqrt{m}}\left(S_k - 2S_{k-m} + 2S_{k-3m} - S_{k-4m}\right),$$

we have for $4m \leq k \leq n$,

$$\left|\frac{Y_k}{\sigma} - \frac{1}{2\sqrt{m}}\Big(W(k) - 2W(k-m) + 2W(k-3m) - W(k-4m)\Big)\right|$$

$$\leq \frac{6}{2\sqrt{m}} \sup_{4m\leq k\leq n} |S_k - W(k)|. \tag{4.3}$$

By (4.2), and noticing that $\frac{\log n}{\sqrt{m}} \to 0$ as $n \to \infty$, we get

$$\lim_{n\to\infty}\left(\max_{4m\le k\le n}\left|\frac{Y_k}{\sigma} - \frac{1}{2\sqrt{m}}\Big(W(k) - 2W(k-m) + 2W(k-3m) - W(k-4m)\Big)\right|\right) = 0, \quad \text{a.s.} \tag{4.4}$$

From Theorem 1, we get

$$\lim_{n\to\infty} P\left\{\sup_{4m\le k\le n}\left|\frac{1}{2\sqrt{m}}\Big(W(k) - 2W(k-m) + 2W(k-3m) - W(k-4m)\Big)\right| \le A_n(x)\right\}$$
$$= \exp\{-2e^{-x}\}, \tag{4.5}$$

where $A_n(x)$ is defined by (2.5). Thus, (2.6) is also true based on (4.3)-(4.5). Theorem 4 is proved.

Notice that under the assumption (4.1), the result of Theorem 3 is also true. We can apply the method of the previous two sections to the case where (4.1) is valid.

Re-examining the condition under which (4.4) is true, we find that with the help of another result of Majors (1976), only is it necessary that $E|e_1|^{2+\delta} < \infty$, where $\delta > 0$. As a result, it is stated as follows:

THEOREM 5. Let $e_1, e_2, \ldots$ be i.i.d. random errors with finite $(2+\delta)$-th moment, where $\delta > 0$, and $m >> n^{2/(2+\delta)}$. Then (2.6) is also true.

5. ESTIMATION OF THE SLOPE CHANGE $\beta_1 - \beta_2$

In order to form a point estimation of the slope change $\beta_1 - \beta_2$, the following procedure is available:

1. Find c such that $|Y_c| = \xi_n = \max_{4m \le k \le n} |Y_k|$,

2. Compute

$$\hat{\beta}_1 - \hat{\beta}_2 = \frac{12n}{c(c^2-1)} \sum_{i=1}^{c} (i - \frac{c+1}{2}) x_i - \frac{12n}{(n-c)\left((n-c)^2-1\right)} \sum_{i=c+1}^{n} (i - \frac{n+c+1}{2}) x_i$$

$$= (F_c' F_c)^{-1} F_c' x. \tag{5.1}$$

The value of $\hat{\beta}_1 - \hat{\beta}_2$ is taken as an estimator of $(\beta_1 - \beta_2)$. It is a LSE of β_1 and β_2 when c is the slope change point. Generally, if c is too near 4m or n, it would imply that the slope change point t_0 is too near 0 or 1, and the samples at our disposal are perhaps not enough to give a reasonable estimate. For an interval estimation of $\beta_1 - \beta_2$, we have the following asymptotic theorem of $\hat{\beta}_1 - \hat{\beta}_2$.

THEOREM 6. Suppose that t_0 is the slope change point and $E|e_1|^{2+\delta} < \infty$ for some $\delta > \frac{2}{3}$, and $m << n^{3/4}$. Then, as $n \to \infty$,

$$\sqrt{\frac{n}{12\sigma^2}\left(t_0^{-3} + (1-t_0)^{-3}\right)^{-1}} \left((\hat{\beta}_1 - \hat{\beta}_2) - (\beta_1 - \beta_2)\right) \xrightarrow{L} N(0,1), \tag{5.2}$$

where $\xrightarrow{L}$ means convergence in law.

Proof. Without losing generality, we assume $\sigma = 1$. Take c such that $|Y_c| = \max_{4m \le j \le n} |Y_j|$. Then, for any $0 < \alpha < 1$ and $\alpha > 0$,

$$
\begin{aligned}
P(nt_0 \le c \le nt_0 + 4m) &= P\left(t_0 \le \frac{c}{n} \le t_0 + \frac{4m}{n}\right) \\
&\ge P\left(\left\{ \sup_{\frac{j}{n} \notin [t_0,\, t_0 + \frac{4m}{n}]} |Y_j| \le c_n(\alpha, d) \right\} \cap \{|Y_c| > c_n(\alpha, d)\}\right) \\
&= P\left(\sup_{\frac{j}{n} \notin [t_0,\, t_0 + \frac{4m}{n}]} |Y_j| \le c_n(\alpha, d)\right) P\left(|Y_c| > c_n(\alpha, d)\right).
\end{aligned}
\tag{5.3}
$$

Using Theorem 5 and slightly modifying the argument of Section 2, we easily prove that

$$\lim_{n \to \infty} P(nt_0 \le c \le nt_0 + 4m) = 1. \tag{5.4}$$

Denote $n_1 = \min\{\ell\colon \frac{\ell}{n} \ge t_0,\ 4m \le \ell \le n - 4m\}$. Not loss of generality, assume $n_1 \le c \le n - 4m$. Because $\hat{\beta}_1 - \hat{\beta}_2$ can be rewritten as

$$
\begin{aligned}
\hat{\beta}_1 - \hat{\beta}_2 &= (0, 1, 0, -1)(F_c'F_c)^{-1}F_c'x \\
&= (0, 1, 0, -1)(F_c'F_c)^{-1}F_c'(F_{n_1}\beta + \varepsilon)
\end{aligned}
\tag{5.5}
$$

by (3.7) and (3.8). So it follows that

$$(\hat{\beta}_1 - \hat{\beta}_2) - (\beta_1 - \beta_2) = (0, 1, 0, -1)(F_c'F_c)^{-1}F_c'(-F_{c-n_1}\beta + \varepsilon), \tag{5.6}$$

where F_{c-n_1} is defined as (3.13). It can be calculated that

$$
(F_c'F_c)^{-1}F_c' = \begin{pmatrix}
(a_{1c} - ka_{2c})e_m' - a_{2c}f_m' & a_{1c}e_k' - a_{2c}f_k' & 0 \\
(na_{2c} - a_{4c}k)e_m' - a_{4c}f_m' & na_{2c}e_k' - a_{4c}f_k' & 0 \\
0 & 0 & b_{1c}e_{n-c}' - b_{2c}g_{n-c}' \\
0 & 0 & -nb_{2c}e_{n-c}' + b_{4c}g_{n-c}'
\end{pmatrix},
\tag{5.7}
$$

where a_{jc}, b_{jc}, $j = 1,2,4$, and e_m, f_m, etc. are defined as (3.8) and (3.11), and $k = c - n_1$. According to (3.3) and (3.13), after replacing $pn - qn_1 \pm 1$ by $pn - qn_1$, where p, q are some integers, we get

$$\begin{aligned} |E\{(\hat{\beta}_1-\hat{\beta}_2) - (\beta_1-\beta_2)\}| &= |-(0,1,0,-1)(F_c'F_c)^{-1}F_c'F_{c-n_1}\beta| \\ &= \frac{6nkn_1}{c^3}(\mu_2-\mu_1) + \frac{k^2(c+2n_1)}{c^3}(\beta_2-\beta_1)| \\ &\le |(\frac{6kn_1}{c^3} + \frac{3k^2c}{c^3})(\beta_2-\beta_1)| \le \frac{4k^2}{c^2}|\beta_2-\beta_1|, \end{aligned} \tag{5.8}$$

and

$$\begin{aligned} \mathrm{Var}\{(\hat{\beta}_1-\hat{\beta}_2) - (\beta_1-\beta_2)\} &= (0,1,0,-1)(F_c'F_c)^{-1}(F_c'F_c)^{-1}(0,1,0,-1)' \\ &= (0,1,0,-1)(F_c'F_c)^{-1}(0,1,0,-1)' \\ &= 12n^2\Big(c^{-1}(c^2-1)^{-1} + (n-c)^{-1}\big((n-c)^2-1\big)^{-1}\Big). \end{aligned} \tag{5.9}$$

Now we verify the three criteria convergenting to standard normal.

1. From the expressions (5.1) and (5.6), we have

$$\begin{aligned} &\mathrm{Var}\Big\{(\hat{\beta}_1-\hat{\beta}_2)-(\beta_1-\beta_2)\Big\}^{-(2+\delta)/2}\Big\{\sum_{i=1}^{c}\Big(\frac{12n}{c(c^2-1)}\Big)^{2+\delta}\Big|i - \frac{c+1}{2}\Big|^{2+\delta}E|e_i|^{2+\delta} + \\ &\qquad \sum_{i=c+1}^{n}\Big(\frac{12n}{(n-c)[(n-c)^2-1]}\Big)^{2+\delta}\Big|i - \frac{n+c+1}{2}\Big|^{2+\delta}E|e_i|^{2+\delta}\Big\} \\ &\le K\,E|e_1|^{2+\delta}\cdot\frac{n^{2+\delta}\Big(c^{-3(2+\delta)+(3+\delta)} + (n-c)^{-3(2+\delta)+(3+\delta)}\Big)}{n^{2+\delta}\Big(c^{-3(2+\delta)/2} + (n-c)^{-3(2+\delta)/2}\Big)} \\ &\le 2K\Big(\max(c,\ n-c)\Big)^{-\delta/2} \le 2Kc^{-\delta/2} \le 2Kt_0^{-\delta/2}n^{-\delta/2} \to 0, \end{aligned} \tag{5.10}$$

where K is a constant.

2. Since $n^{3/4} \gg k$, we get

$$\lim_{n\to\infty} \frac{|E\{(\hat\beta_1-\hat\beta_2)-(\beta_1-\beta_2)\}|}{\sqrt{\mathrm{Var}\{(\hat\beta_1-\hat\beta_2)-(\beta_1-\beta_2)\}}} \le \lim_{n\to\infty} \frac{4k^2}{c^2}\,|\beta_2-\beta_1|\cdot(12n^2c^{-3})^{-1/2}$$

$$\le \lim_{n\to\infty} \frac{2k^2}{\sqrt{3}\,t_0\,n^{3/2}} = 0. \tag{5.11}$$

3. It is easy to see that

$$12n^3\Big(c^{-1}(c^2-1)^{-1} + (n-c)^{-1}\big((n-c)^2-1\big)^{-1}\Big) \to 12\Big(t_0^{-3} + (1-t_0)^{-3}\Big). \tag{5.12}$$

Combining (5.10)-(5.12), we prove this theorem.

Notice that $\hat t_0 = (c-2m)/n$ is a consistent estimator of t_0. (Of course, only when $\beta_1 - \beta_2 \neq 0$, t_0 is well-defined.) In Section 3, we have introduced a consistent estimator $\hat\sigma_n$ of σ. Substituting $\hat t_0$ for t_0 and $\hat\sigma_n$ for σ, we can further get this result.

THEOREM 7. Suppose that the conditions of Theorem 6 are satisfied. We then have

$$\left\{\frac{n}{12\hat\sigma_n^2}\Big(\hat t_0^{-3} + (t-\hat t_0)^{-3}\Big)^{-1}\right\}^{1/2}\Big\{(\hat\beta_1-\hat\beta_2)-(\beta_1-\beta_2)\Big\} \xrightarrow{L} N(0,1). \tag{5.13}$$

as $n \to \infty$.

When $\beta_1 = \beta_2$, t_0 has no meaning, but the statistic $\hat t_0$ is still well defined. It is not known whether or not (5.13) is true for $\beta_1 = \beta_2$, and (5.13) cannot be used to make a test for the hyperthesis $\beta_1 = \beta_2$. However, (5.13) can be utilized to form a confidence interval of $(\beta_1 - \beta_2)$ if we know $\beta_1 \neq \beta_2$ a priori or the null hypothesis (2.13) is rejected.

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